Pseudo Magic Squares

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Abstract

A magic square of order n is an $n \times n$ square (matrix) whose entries are distinct nonnegative integers such that the sum of the numbers of any row and column is the same number, the magic constant. In this paper we introduce the concept of pseudo magic squares, *i.e.*, magic squares defined over the ring of integers, without the restriction of distinct numbers. Additionally, we generalize this new concept by introducing a group (ring) structure over it. This new approach can provide useful tools in order to find new non-isomorphic pseudo magic squares.

1 Introduction

The concept of magic squares is well known in literature [2, 3, 5]. The Loh-Shu magic square

$$
\begin{bmatrix}
4 & 9 & 2 \\
3 & 5 & 7 \\
8 & 1 & 6
\end{bmatrix}
$$

is the oldest known magic square and its invention is attributed to Fuh-Hi (2858-2738 b.C.) [5]. There exist interesting papers available in the literature dealing with constructions of magic squares [1–4]. Recently, Xin [7] have constructed all magic squares of order three. In most works available in literature, the approach adopted is to apply combinatorial methods to compute more classes of new (non-isomorphic) magic squares.

In this paper we introduce the concept of pseudo magic square (PMS). We show that a PMS have a natural group structure. Additionally, we generalize this new concept in order to obtain a generic magic square (GMS), which is derived from a arbitrary group (ring). The structure of a group (ring) induces a group (ring) structure in the set of GMS's. Based on these facts, one can see that our approach is quite different of the ones available in literature.

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The paper is organized as follows. In Section 2, we fix the notation and also define the concept of pseudo magic squares. In Section 3, we present the contributions of this paper: some properties of PMS are shown as well as the introduction of the concept of generic magic squares with corresponding properties are exhibited. In Section 4, a brief summary of this paper is given.

2   Preliminaries

Notation. Throughout this paper, $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{N} - \{0\} = \mathbb{N}^*$, $\mathbb{Z}$ is the set of integers. The set of square matrices of order n with entries in $\mathbb{Z}$ is denoted by $\mathbb{M}_n(\mathbb{Z})$; the cardinality of a set S is denoted by $|S|$.

Definition **2.1** *Let $n \in \mathbb{N}^*$. A pseudo magic square of order n, denoted by $A_n^\square$, is an element of $\mathbb{M}_n(\mathbb{Z})$ such that the sum of the numbers of any row and column is the same number $c_{A_n^\square}$, the pseudo magic constant (constant, for short). The set of PMS of order n is denoted by $\mathcal{P}_n^\square$.*

Remark **2.1** *To avoid stress of notation, we do not distinguish between an $n \times n$ matrix or an $n \times n$ square.*

Example **2.1** *As an example, we have a PMS of order 4 given by*

$$
\begin{bmatrix}
-5 & -5 & -5 & -5 \\
-5 & -5 & -5 & -5 \\
-5 & -5 & -5 & -5 \\
-5 & -5 & -5 & -5
\end{bmatrix}.
$$

3   The Results

In this section we present the results of this work. We start by showing that the set of PMS endowed with the sum of matrices is a group:

Theorem **3.1** *The ordered pair $(\mathcal{P}_n^\square, +)$ is an abelian group, where the operation $+$ means the addition of matrices.*

Proof: If $A_n^\square$ and $B_n^\square$ are PMS of constants $c_{A_n^\square}$ and $c_{B_n^\square}$, respectively, then $A_n^\square + B_n^\square$ is a PMS with constant $c_{A_n^\square} + c_{B_n^\square}$, so $\mathcal{P}_n^\square$ is closed. It is clear that the null PMS of order n is the identity element. The associativity follows trivially of the associativity of the (additive) group $(\mathbb{Z}, +)$. The same applies to commutativity. It is clear that the inverse of a PMS $A_n^\square$ of constant $c_{A_n^\square}$ is the PMS $[-A]_n^\square$ of constant $-c_{A_n^\square}$. $\qquad\square$

Corollary **3.2** *$(\mathcal{P}_n^\square, +)$ is a subgroup of $(\mathbb{M}_n(\mathbb{Z}), +)$.*

Proof: Straightforward. □

It is possible to construct new PMS's from old ones. In fact, the first method to construct new PMS's is by means of the direct sum structure (see Theorem 3.3). It is clear that several structures can be defined over PMS's as well, but such structures will be considered in other paper.

Theorem **3.3** *Let $A_n^{\square}$ and $B_n^{\square}$ be two PMS's. Define the direct sum $A_n^{\square} \oplus B_n^{\square}$ as*

$$\left[\begin{array}{cc} A_n^{\square} & B_n^{\square} \\ B_n^{\square} & A_n^{\square} \end{array} \right] \in \mathbb{M}_{2n}(\mathbb{Z}).$$

Then the set $\mathcal{P}_n^{\square} \oplus \mathcal{P}_n^{\square}$ of direct sums endowed with addition of matrices is an abelian group.

Proof: The direct sum is well-defined. Let $A_n^{\square}$ and $B_n^{\square}$ be two PMS's of constants $c_{A_n^{\square}}$ and $c_{B_n^{\square}}$, respectively. Then $A_n^{\square} \oplus B_n^{\square}$ is a PMS of order $2n$ and constant $c_{A_n^{\square}} + c_{B_n^{\square}}$. If $C_n^{\square}$ and $D_n^{\square}$ are also PMS's of constants $c_{C_n^{\square}}$ and $c_{D_n^{\square}}$, respectively, then its sum $[A_n^{\square} \oplus B_n^{\square}] + [C_n^{\square} \oplus D_n^{\square}]$ is also a PMS of order $2n$ and constant $c_{A_n^{\square}} + c_{B_n^{\square}} + c_{C_n^{\square}} + c_{D_n^{\square}}$. Then $\mathcal{P}_n^{\square} \oplus \mathcal{P}_n^{\square}$ is closed. The remaining properties follow similarly as in the proof of Theorem 3.1. □

Corollary **3.4** *$(\mathcal{P}_n^{\square} \oplus \mathcal{P}_n^{\square}, +)$ is a subgroup of $(\mathbb{M}_{2n}(\mathbb{Z}), +)$.*

Proof: Straightforward. □

Let $A_n^{\square}$ be a PMS of order n. We can multiply each entry of $A_n^{\square}$ by an integer k. Then the resulting square is also a PMS square. Similarly, by adding an integer to each entry we obtain again a PMS. Moreover, one can define the tensor product of PMS's. Note that such ideas can be explored in order to generate several properties as well as new structures for PMS's.

Next, we introduce the concept of generic magic squares:

Definition **3.1** *Let $(G, \star)$ be an abelian group. A generic magic square, of order n, derived from $(G, \star)$, is an $n \times n$ matrix with entries in G such that for any row and any column, the result of the operation among the row(column) elements corresponds to the same element, the generic magic constant (constant, for short).*

We denote the set of GMS's of order n over G by $[\mathcal{G}g]_{G_n}^{\square}$.

Remark **3.5** *It is clear that $[\mathcal{G}g]_{G_n}^{\square} \neq \emptyset$. In fact, if e_G is the identity of G, then*

$$[E]_{G_n}^{\square} = \left[\begin{array}{cccc} e_G & e_G & \cdots & e_G \\ e_G & e_G & \cdots & e_G \\ \vdots & \vdots & \cdots & \vdots \\ e_G & e_G & \cdots & e_G \end{array} \right] \in \mathbb{M}_n(G) \text{ is a GMS over } G.$$

Theorem **3.6** *The ordered pair* $([\mathcal{G}g]^{\square}_{G_n}, \star_p)$, *where* $\star_p$ *denotes the group operation component-wise in* $\mathbb{M}_n(G)$, *is an abelian group.*

Proof: It is clear that the set is closed. The identity is the GMS $[E]^{\square}_{G_n}$. The remaining properties follows similarly as in the proof of Theorem 3.1. The detailed proof will be provided in other paper. $\square$

By lack of space, we will not investigate in this paper the structures of this interesting group, *i.e.*, homomorphisms and isomorphisms, group actions, properties. However, these subjects of research will be investigated in a future paper.

Consider now a commutative ring with unit $(R, +, \cdot)$. Similarly as was done with the group structure, the ring $(R, +, \cdot)$ induces a natural ring structure in the set os GMS as follows:

Definition **3.2** *Let* $(R, +, \cdot)$ *be an commutative ring with unit. A generic magic square of order n, derived from* $(R, +, \cdot)$, *is an* $n \times n$ *matrix with entries in* R *such that:*
(1) For any row and any column, the result of the operation $+$ *among the row(column) elements corresponds to the same element, the generic additive magic constant (additive constant, for short);*
(2) For any row and any column, the result of the operation $\cdot$ *among the row(column) elements corresponds to the same element, the generic multiplicative magic constant (multiplicative constant, for short).*

We denote by $[\mathcal{G}r]^{\square}_{R_n}$ the set of GMS's of order n over R. Note that the additive constant can be different from the multiplicative constant. Analogously to groups, the set $[\mathcal{G}r]^{\square}_{R_n}$ is not empty.

Theorem 3.7, given in the following, states that the set $[\mathcal{G}r]^{\square}_{R_n}$ endowed with two component-wise operations form a commutative ring with unit. We give only a sketch of the proof.

Theorem **3.7** *The ordered triple* $([\mathcal{G}r]^{\square}_{R_n}, +_p, \cdot_p)$, *where* $+_p$ *and* $\cdot_p$ *denote the ring operations component-wise in* $\mathbb{M}_n(R)$, *is a commutative ring with unit.*

Proof: It is easy to see that the set $[\mathcal{G}r]^{\square}_{R_n}$ is closed under the operations $+_p$ and $\cdot_p$. The commutativity, associativity and distributivity of such operations follow from the commutativity, associativity and distributivity, respectively, of the operations $+$ and $\cdot$ of the ring. The identity and the unit of $[\mathcal{G}r]^{\square}_{R_n}$ are obvious from the context. $\square$

Given a GMS $([\mathcal{G}r]^{\square}_{R_n}, +_p, \cdot_p)$, we can operate (the operation considered here is $\cdot$ of the ring) a fixed element of the ring $r \in R$ with each entry in $[\mathcal{G}r]^{\square}_{R_n}$. It is easy to see that the resulting matrix is also a GMS. In this context, one can derive more algebraic structures over $([\mathcal{G}r]^{\square}_{R_n}, +_p, \cdot_p)$.

Definition **3.3** *A matroid M is an ordered pair* $(S, \mathcal{I})$ *consisting of a finite set S and a collection* $\mathcal{I}$ *of subsets of S satisfying the following three conditions:*
(I.1) $\emptyset \in \mathcal{I}$;

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(I.2) If $I \in \mathcal{I}$ and $I' \subset I$, then $I' \in \mathcal{I}$;
(I.3) If I_1, $I_2 \in \mathcal{I}$ and $\mid I_1 \mid < \mid I_2 \mid$, then there exists an element $e \in I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

It is well known that the concept of matroid generalizes several algebraic structures. In this context, we describe here, how to induce a matroid structure on the set $[\mathcal{G}r]_{R_n}^{\square}$. In particular, if we consider the class of vectorial matroid (see [6]), interesting results can be proved.

4 Summary

In this paper, we have introduced the concept of pseudo magic squares and generic magic squares. Moreover, we have shown that the set of generic (pseudo) magic squares has a group (ring) structure. Additionally, some properties of PMS and GMS as well as constructions of new PMS's from old ones have also been presented. From the content exhibited in this paper, it can be observed that there exist much work with respect to our new approach to be done. Other interesting possibility is the introduction of a topological structure in GMS's.

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